

CLASSICAL GAUGE THEORY OF GRAVITY¹

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The classical theory of gravity is formulated as a gauge theory on a frame bundle with spontaneous symmetry of breaking caused by the existence of Dirac fermionic fields. The pseudo-Riemannian metric (tetrad field) is the corresponding Higgs field. We consider two variants of this theory. In the first variant, gravity is represented by the pseudo-Riemannian metric as in general relativity theory; in the second variant, it is represented by the effective metric as in the Logunov relativistic theory of gravity. The configuration space, Dirac operator, and Lagrangians are constructed for both variants.

Key words: gravity, gauge field, Higgs field, spinor field

1 Introduction

The first gauge model of gravity [1] was built just two years after birth of the gauge theory itself. But the initial attempts to construct the gauge theory of gravity by analogy with the gauge models of internal symmetries encountered a serious difficulty – establishing the gauge status of the metric (tetrad) field and choosing the gauge group. In [1], gravity was described in the framework of the gauge model of the Lorentz group, but tetrad fields were introduced arbitrarily. To eliminate this drawback, representing tetrad fields as gauge fields of the translation group was attempted [2]-[9], but to no effect. At first, the canonical lift of vector fields on a manifold X to the tangent bundle TX [2], [3], which is actually the generator of general covariant transformations of TX , was considered as generators of the translation gauge group. A horizontal lift of vector fields on X using a linear connection on $TX \rightarrow X$ was proposed for the same role [4]. The tetrad field was later identified with the translation part of an affine connection on the tangent bundle $TX \rightarrow X$. Any such connection is a sum of a linear connection and a tangent-valued form

$$\Omega = \Omega_\lambda^\alpha(x) dx^\lambda \otimes \partial_\alpha, \quad (1)$$

but this form and the tetrad field are two different mathematical objects [10], [11]. We note that form (1) keeps appearing in one of the variants of the gauge theory of gravity as

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a nonholonomic frame [12]. At the same time, it turned out that translational connections describe distortion in the gauge theory of dislocations [13], [14] and in the analogous gauge model of the "fifth force" [11], [15].

Difficulties of constructing the gauge theory of gravity by analogy with the gauge theory of internal symmetries resulted from the gauge transformations in these theories belonging to different classes. In the case of internal symmetries, the gauge transformations are just vertical automorphisms of the principal bundle $P \rightarrow X$ leaving its base X fixed. On the other hand, gravity theory is built on the principal bundle LX of the tangent frames to X and tensor bundles to X associated with it. They belong to the category of bundles $T \rightarrow X$ for which diffeomorphisms f of the base X canonically generate automorphisms \tilde{f} of the bundle T . These automorphisms are called holonomic automorphisms or general covariant transformations. All gravitational Lagrangians are constructed to be invariant under general covariant transformations, which are therefore gauge transformations of gravity theory. Nonholonomic (e.g., vertical) automorphisms of the principal bundle LX can also be considered, but most gravitational Lagrangians are not left invariant under these transformations.

In the persistent attempts to represent the terad field as a gauge field, the presence of Higgs fields in addition to the matter and gauge fields in the gauge theory with spontaneous symmetry breaking was overlooked. Spontaneous symmetry breaking is a quantum effect when the vacuum is not invariant under the transformation group. In terms of the classical gauge theory, spontaneous symmetry breaking occurs if the structure group G of the principal bundle $P \rightarrow X$ is reducible to a closed subgroup H , i.e., there exists a principal subbundle of the bundle P with the structure group H [10], [16]-[18]. Such a reduction occurs if and only if the quotient bundle $P/H \rightarrow X$ admits a global section. Moreover, there exists a one-to-one correspondence between reduced subbundles $P^h \subset P$ with the structure group H and global sections h of the quotient bundle $P/H \rightarrow X$ [19]. The latter are treated as classical Higgs fields.

The idea of the pseudo-Riemannian metric as a Higgs-Goldstone field appeared while constructing nonlinear (induced) representations of the group $GL(4, \mathbf{R})$, of which the Lorentz group is a Cartan subgroup [20], [21]. When the gauge theory was formulated in terms of bundles, the very definition of the pseudo-Riemannian metric on a manifold X as a global section of the quotient bundle

$$\Sigma_{PR} = LX/SO(1, 3) \rightarrow X \quad (2)$$

led to its physical interpretation as a Higgs field responsible for the spontaneous space-time symmetry breaking [10, [22], [23]. The geometric principle of equivalence postulating the existence of a reference frame in which Lorentz invariants are defined on the whole space-time manifold is the theoretical justification of such symmetry breaking [10]. In terms of bundles, this means that the structure group $GL(4, \mathbf{R})$ of the frame bundle and

the bundles associated with it is reduced to the Lorentz group.

The physical reason for space-time symmetry breaking is the existence of Dirac fermionic matter, whose symmetry group is the universal two-sheeted covering $L_s = SL(2, \mathbf{C})$ of the proper Lorentz group $L = SO^\uparrow(1, 3)$. The Dirac spin structure on a space-time manifold X is defined as a pair (P^h, z_s) of a principal bundle $P^h \rightarrow X$ with the structure group L_s and the fiberwise morphism

$$z_s : P^h \rightarrow LX \quad (3)$$

to the frame bundle LX [24], [25]. Any such morphism can be factored using a morphism

$$z_h : P^h \rightarrow L^h X, \quad (4)$$

where $L^h X$ is a principal subbundle of the frame bundle LX with the structure group being the Lorentz group L .

We assume in what follows that X is a four-dimensional oriented (simply connected, smooth, separate, locally compact, countably infinite, i.e., paracompact) manifold with a coordinate atlas $\{(U_\zeta, x^\lambda)\}$ and that LX is the bundle of oriented tangent frames to X with the structure group $GL_4 = GL^+(4, \mathbf{R})$. Then there exists a one-to-one correspondence between the abovementioned reduced bundles $L^h X$ with the structure group L and global sections h of the quotient bundle

$$\Sigma = LX/L \rightarrow X \quad (5)$$

with a typical fiber GL_4/L . In other words, the restriction of the bundle $LX \rightarrow \Sigma$ to $h(X) \subset \Sigma$ is isomorphic to $L^h X$. Bundle (5) is a two-sheeted covering of the bundle Σ_{PR} given by (2). Its global section h is called the tetrad field. It is represented by a collection of local sections $h_\mu^a(x)$ of the reduced subbundle $L^h X$, which are called tetrad functions. Any tetrad field uniquely determines a pseudo-Riemannian metric on X such that the well-known identity

$$g_{\mu\nu} = h_\mu^a h_\nu^b \eta_{ab}, \quad (6)$$

holds, where η is the Minkowski metric. A Dirac spin structure on the space-time manifold thus uniquely determines a pseudo-Riemannian metric on it.

The existence of a Dirac spin structure also imposes severe restrictions on the topology of the space-time manifold X [26], [27]. Because compact space-time does not satisfy any causality principle, we assume in what follows that the manifold X is not compact. Then it is parallelizable, i.e., the frame bundle $LX \rightarrow X$ is trivial. In this case, all spin structures P^h on X are mutually isomorphic although this isomorphism is not canonical.

Let $S^h \rightarrow X$ denote the spinor bundle associated with the principal bundle P^h . Its sections describe Dirac fermionic fields in the presence of a tetrad field h . The problem is that in the presence of different tetrad fields, the fermionic fields are represented by

sections of different bundles with no canonical isomorphisms between them. Moreover, to define the Dirac operator on the sections of the bundle S^h , we must specify a representation

$$dx^\lambda \mapsto \gamma_h(dx^\lambda) = h_a^\lambda(x) \gamma^a \quad (7)$$

of coframes $\{dx^\lambda\}$ tangent to the space-time manifold X by Dirac matrices. But these representations are not equivalent for different tetrad fields h . The Higgs nature of the classical tetrad field manifests itself here. A Dirac field on the space-time manifold can only be considered paired with a definite tetrad field.

In particular, because morphism (3) is factored using morphism (4), the whole existing fermionic matter represented by sections of some spinor bundle over X uniquely fixes the Lorentz structure $L^h X$ and tetrad field h (pseudo-Riemannian metric g) on X , this being independent of the fermionic field dynamics. Therefore, this tetrad field (pseudo-Riemannian metric) is not determined by any differential equations and is a background field. Consequently, it cannot be identified with the gravitational field, and a background geometry is present in the gravity theory. This considerations motivate us to regard two variants of the gauge theory of gravity.

In the first variant [11], [28]-[31], the gravitational field is identified with the tetrad field, and we consider the composite bundle

$$S \rightarrow \Sigma \rightarrow X, \quad (8)$$

where $S \rightarrow \Sigma$ is the spinor bundle associated with the principal bundle $LX \rightarrow \Sigma$ with the structure group being the Lorentz group L . The idea of this construction is that for any section h of the tetrad bundle $\Sigma \rightarrow X$, the restriction of the bundle $S \rightarrow \Sigma$ to $h(X) \subset \Sigma$ is isomorphic to the spinor bundle $S^h \rightarrow X$, whose sections are fermionic fields in the presence of a tetrad gravitational field h . Therefore, sections of bundle (8) describe the totality of fermionic and gravitational fields.

The second variant [29], [32] uses the fact that if Q is a group bundle associated with LX , then there exists a morphism

$$\rho : Q \times \Sigma \rightarrow \Sigma \quad (9)$$

that assigns to any tetrad field h a background tetrad field h_0 and a section q of the bundle $Q \rightarrow X$, the latter section being identified with the gravitational field. In this approach, Dirac fermionic fields are described by sections of the bundle $S^{h_0} \rightarrow X$, and their dynamics in the presence of gravitational field q appears as motion in the effective tetrad field \tilde{h} .

We define the configuration space, Dirac operator, and Lagrangians for both variants of the gauge theory of gravity.

2 Lorentz structure

As already stated, the theory of gravity is built as a gauge theory on the principal bundle $LX \rightarrow X$ of oriented frames tangent to the space-time X with the structure group GL_4 . We say a few words about this bundle. Any element of it can be represented in the form $H_a = H_a^\mu \partial_\mu$, where $\{\partial_\mu\}$ is the set of holonomic bases of the tangent bundle TX and H_a^μ are the matrix elements of the representation of the group GL_4 in \mathbf{R}^4 . The latter serve as coordinates on the bundle LX with the transition functions

$$H_a'^\mu = \frac{\partial x'^\mu}{\partial x^\lambda} H_a^\lambda.$$

In these coordinates, the right action of the structure group GL_4 on LX becomes

$$R_g : H_a^\mu \mapsto H_b^\mu g^b{}_a, \quad g \in GL_4. \quad (10)$$

The frame bundle LX is equipped with a canonical \mathbf{R}^4 -valued 1-form

$$\theta_{LX} = H_\mu^a dx^\mu \otimes t_a, \quad (11)$$

where $\{t_a\}$ is a fixed basis for \mathbf{R}^4 and H_μ^a are elements of the inverse matrix.

As in any gauge theory, connections on the principal frame bundle LX are the gauge fields in the gravity theory. They are in one-to-one correspondence with linear connections K on the tangent bundle TX (or just on X). In the holonomic coordinates $(x^\lambda, \dot{x}^\lambda)$ on TX , these connections are given by the tangent-valued forms

$$K = dx^\lambda \otimes (\partial_\lambda + K_\lambda{}^\mu{}_\nu \dot{x}^\nu \partial_\mu) \quad (12)$$

and are represented by sections of the quotient bundle

$$C = J^1 LX / GL_4 \rightarrow X, \quad (13)$$

where $J^1 LX$ is the manifold of first-order jets of sections of the bundle $LX \rightarrow X$ [31]. The bundle of connections C is equipped with the coordinates $(x^\lambda, k_\lambda{}^\nu{}_\alpha)$ such that the coordinates $k_\lambda{}^\nu{}_\alpha \circ K = K_\lambda{}^\nu{}_\alpha$ of any section K are the coefficients of corresponding linear connection (12). We stress that the bundle of connections C given by (13) is not associated with the frame bundle but also admits general covariant transformations.

As already mentioned, the metric (tetrad) field is introduced in the gauge theory of gravity by specifying the Lorentz structure. The subbundle $L^h X$ of the frame bundle LX with the structure group L , where h is a global section of quotient bundle (5), is called the Lorentz structure on the space-time manifold X .

The tetrad field h determines the atlas $\Psi^h = \{(U_\zeta, z_\zeta^h)\}$ of the frame bundle LX such that local sections z_ζ^h of the bundle LX take values in the Lorentz subbundle $L^h X$ and

have Lorentz transition functions. They are called the tetrad functions and have the form $h_a^\mu = H_a^\mu \circ z_\zeta^h$. Tetrad functions induce the local tetrad form

$$z_\zeta^{h*} \theta_{LX} = h_\lambda^a dx^\lambda \otimes t_a$$

on X , where θ_{LX} is canonical form (11) on LX . This tetrad form in turn determines tetrad coframes

$$h^a = h_\mu^a(x) dx^\mu, \quad x \in U_\zeta,$$

on the cotangent bundle T^*X of X . In particular, identity (6) becomes $g = \eta_{ab} h^a \otimes h^b$. The metric g is thus reduced to the Minkowski metric with respect to the Lorentz atlas Ψ^h and is an example of Lorentz invariants in the geometric equivalence principle.

We now consider gauge fields in the presence of the Lorentz structure. Connections on the Lorentz bundle $L^h X$ have the form

$$A_h = dx^\lambda \otimes (\partial_\lambda + \frac{1}{2} A_\lambda^{ab} \varepsilon_{ab}), \quad (14)$$

where $\varepsilon_{ab}{}^c{}_d = \eta_{ad} \delta_b^c - \eta_{bd} \delta_a^c$ are the generators of Lorentz group. We call them Lorentz connections. Taking the property of equivariance with respect to right action (10) of the structure group on the principal bundle into account, we can extend any Lorentz connection on $L^h X$ to a connection on the frame bundle LX thus determining a linear connection K (see (12)) on X with coefficients

$$K_\lambda^\mu{}_\nu = h_\nu^k \partial_\lambda h_k^\mu + \eta_{ka} h_b^\mu h_\nu^k A_\lambda^{ab}. \quad (15)$$

We can formulate the converse statement. According to the known theorem [19], a linear connection on X is a Lorentz connection if and only if its holonomy group reduces to the Lorentz group. At the same time, it can be shown [29], [31] that on any Lorentz subbundle $L^h X$, any linear connection K given by (12) determines a Lorentz connection A_h given by (14) with the coefficients

$$A_\lambda^{ab} = \frac{1}{2} (\eta^{kb} h_\mu^a - \eta^{ka} h_\mu^b) (\partial_\lambda h_k^\mu - h_k^\nu K_\lambda^\mu{}_\nu). \quad (16)$$

This allows using general connections independent of any spin structure in a gravity theory with fermionic fields.

3 Universal spin structure

Because the first homotopic group of the group space GL_4 equals \mathbf{Z}_2 , there exists a universal covering group \widetilde{GL}_4 of the group GL_4 such that the diagram

$$\begin{array}{ccc} \widetilde{GL}_4 & \longrightarrow & GL_4 \\ \uparrow & & \uparrow \\ L_s & \xrightarrow{z_L} & L \end{array}$$

is commutative. The group \widetilde{GL}_4 has a spinor representation whose elements, called "world" spinors, were proposed for describing fermions in gravity theory [12]. But this representation is infinite-dimensional. We therefore choose another way.

We consider the two-sheeted covering $\widetilde{LX} \rightarrow X$ of the frame bundle LX , which is a principal bundle with the structure group \widetilde{GL}_4 [25], [33], [34]. Because $\Sigma = \widetilde{LX}/L_s$, the bundle

$$\widetilde{LX} \rightarrow \Sigma \tag{17}$$

is a principal bundle with the structure group L_s . Therefore, the commutative diagram

$$\begin{array}{ccc} \widetilde{LX} & \xrightarrow{\widetilde{z}} & LX \\ & \searrow \swarrow & \\ & \Sigma & \end{array} \tag{18}$$

is valid, which gives the spin structure on the tetrad bundle Σ . This spin structure is unique and has the following property [29]-[31]. For any global section h of the bundle $\Sigma \rightarrow X$ given by (5), the restriction of principal bundle (17) to $h(X) \subset \Sigma$ is isomorphic to the principal bundle P^h given by (3) with the structure group L_s . Therefore, spin structure (18) is said to be universal.

We consider the spinor bundle $S \rightarrow \Sigma$ associated with (17). Similarly to the preceding, if h is a section of the bundle $\Sigma \rightarrow X$, then the restriction h^*S of the bundle $S \rightarrow \Sigma$ to $h(X) \subset \Sigma$ is a subbundle of the composite bundle $S \rightarrow X$ given by (8) and is isomorphic to the spinor bundle S^h associated with P^h . We note that $S \rightarrow X$ is not a spinor bundle. In particular, holonomic automorphisms of the frame bundle LX unambiguously extend to \widetilde{LX} and induce general covariant transformations of the bundle $S \rightarrow X$ [28]-[31].

We construct the Dirac operator on the spinor bundle $S \rightarrow \Sigma$ such that its restriction to the subbundle S^h of the bundle $S \rightarrow X$ reproduces the Dirac operator of fermionic fields in the presence of a tetrad field h and a general linear connection K on X . We recall that connections on the spinor bundle $S^h \rightarrow X$ are associated with connections on the principal bundle P^h and are in one-to-one correspondence with Lorentz connections

on $L^h X$. It follows that any Lorentz connection A_h given by (14) determines the spinor connection

$$A_h = dx^\lambda \otimes (\partial_\lambda + A^{ab}{}_\lambda L_{ab}{}^A{}_B y^B \partial_A), \quad L_{ab} = \frac{1}{4}[\gamma_a, \gamma_b], \quad (19)$$

on the bundle S^h equipped with the coordinates (x^λ, y^A) relative to the Lorentz atlas $\Psi^h = \{h_\lambda^a\}$ of the bundle $L^h X$ extended to P^h . Consequently, an arbitrary linear connection K given by (12) on X induces a Lorentz connection on $L^h X$ with coefficients (16) and correspondingly a spinor connection

$$K_h = dx^\lambda \otimes [\partial_\lambda + \frac{1}{4}(\eta^{kb} h_\mu^a - \eta^{ka} h_\mu^b)(\partial_\lambda h_k^\mu - h_k^\nu K_{\lambda}{}^\mu{}_\nu) L_{ab}{}^A{}_B y^B \partial_A] \quad (20)$$

on S^h [28], [31], [35]. Spinor connection (19) determines the covariant differential

$$D : J^1 S^h \rightarrow T^* X \otimes S^h, \quad D = (y_\lambda^A - A^{ab}{}_\lambda L_{ab}{}^A{}_B y^B) dx^\lambda \otimes \partial_A, \quad (21)$$

on S^h , where $J^1 S^h$ with coordinates $(x^\lambda, y^A, y_\lambda^A)$ is the manifold of jets of sections of the fibre bundle $S^h \rightarrow X$. On the bundle S^h , the covariant differential D given by (21) in composition with representation (7) determines the Dirac operator

$$\begin{aligned} \Delta_h &= \gamma_h \circ D : J^1 S^h \rightarrow T^* X \otimes S^h \rightarrow S^h, \\ y^A \circ \Delta_h &= h_a^\lambda \gamma^{aA}{}_B (y_\lambda^B - \frac{1}{2} A^{ab}{}_\lambda L_{ab}{}^A{}_B y^B) \end{aligned} \quad (22)$$

of fermionic fields in the presence of a background tetrad field h and a general linear connection K .

We now turn to the bundle $S \rightarrow \Sigma$ and spinor bundle Y over the product $\Sigma \times C$ induced from it, where C is the bundle of general linear connections (13). The bundle Y is equipped with the coordinates $(x^\lambda, y^A, \sigma_\mu^a, k_{\lambda}{}^\mu{}_\nu)$, where $(x^\lambda, \sigma_\mu^a)$ are coordinates on the tetrad bundle Σ such that the coordinates $\sigma_\mu^a \circ h$ of any section h are tetrad functions h_μ^a . The manifold of jets $J^1 Y$ of the bundle $Y \rightarrow \Sigma \times C \rightarrow X$ is the configuration space of the complete system of fermionic, tetrad, and gauge fields in the gauge theory of gravity where the tetrad field and corresponding pseudo-Riemannian metric are identified with the gravitational field. Dirac operator on this configuration space has the form [28]-[31]

$$\begin{aligned} \Delta_Y &= \gamma_\Sigma \circ \widetilde{D} : J^1 Y \rightarrow T^* X \otimes_\Sigma S \rightarrow S, \\ y^B \circ \Delta_Y &= \sigma_a^\lambda \gamma^{aB}{}_A [y_\lambda^A - \frac{1}{4}(\eta^{kb} \sigma_\mu^a - \eta^{ka} \sigma_\mu^b)(\sigma_{\lambda k}^\mu - \sigma_k^\nu k_{\lambda}{}^\mu{}_\nu) L_{ab}{}^A{}_B y^B]. \end{aligned} \quad (23)$$

It satisfies the requirement stated above and coincides with Dirac operator (22) when restricted to $h(X) \times K(X) \subset \Sigma \times C$ for a background tetrad field h and a linear connection K .

The Lagrangian of the gauge theory of gravity on the configuration space J^1Y can be chosen in the form of a sum

$$L = L_D + L_{AM} \quad (24)$$

of the Dirac Lagrangian

$$\begin{aligned} L_D = & \left\{ \frac{i}{2} \sigma_q^\lambda [y_A^+ (\gamma^0 \gamma^q)^A{}_B (y_\lambda^B - \frac{1}{4} (\eta^{kb} \sigma_\mu^a - \eta^{ka} \sigma_\mu^b) (\sigma_{\lambda k}^\mu - \sigma_k^\nu k_\lambda^\mu{}_\nu) L_{ab}{}^B{}_C y^C) - \right. \\ & (y_{\lambda A}^+ - \frac{1}{4} (\eta^{kb} \sigma_\mu^a - \eta^{ka} \sigma_\mu^b) (\sigma_{\lambda k}^\mu - \sigma_k^\nu k_\lambda^\mu{}_\nu) y_C^+ L_{ab}{}^{+C}{}_A (\gamma^0 \gamma^q)^A{}_B y^B] - \\ & \left. m y_A^+ (\gamma^0)^A{}_B y^B \right\} \sqrt{|\sigma|}, \quad \sigma = \det(\sigma_{\mu\nu}), \end{aligned} \quad (25)$$

and the Lagrangian

$$L_{AM}(R_{\lambda\mu}{}^\alpha{}_\beta, \sigma^{\mu\nu}), \quad \sigma^{\mu\nu} = \sigma_a^\mu \sigma_b^\nu \eta^{ab},$$

of the affine-metric theory expressed via the curvature tensor of the linear connection

$$R_{\lambda\mu}{}^\alpha{}_\beta = k_{\lambda\mu}{}^\alpha{}_\beta - k_{\mu\lambda}{}^\alpha{}_\beta + k_\mu{}^\alpha{}_\varepsilon k_\lambda{}^\varepsilon{}_\beta - k_\lambda{}^\alpha{}_\varepsilon k_\mu{}^\varepsilon{}_\beta. \quad (26)$$

It is easy to see that

$$\frac{\partial \mathcal{L}_D}{\partial k_\lambda{}^\mu{}_\nu} + \frac{\partial \mathcal{L}_D}{\partial k_\nu{}^\mu{}_\lambda} = 0.$$

Therefore, Dirac Lagrangian (25) depends only on the torsion tensor $k_\lambda{}^\mu{}_\nu - k_\nu{}^\mu{}_\lambda$ of the linear connection. Moreover, full Lagrangian (24) is invariant under nonholonomic gauge transformations

$$k_\lambda{}^\mu{}_\nu \rightarrow k_\lambda{}^\mu{}_\nu + V_\lambda \delta_\nu^\mu,$$

i.e., has the so-called projective freedom.

4 Goldstone gravitational field

We consider the group bundle $Q \rightarrow X$ associated with LX whose typical fiber is the group GL_4 acting on itself via the adjoint representation. It is equipped with the coordinates $(x^\lambda, q^\lambda{}_\mu)$ as a subbundle of the tensor bundle $TX \otimes T^*X$ and admits a canonical section $q_0(x) = \partial_\mu \otimes dx^\mu$. The canonical left action Q on any bundle associated with LX is given. In particular, action (9) on the tetrad bundle Σ has the form

$$\rho : (x^\lambda, q^\lambda{}_\mu, \sigma_a^\mu) \mapsto (x^\lambda, q^\lambda{}_\mu \sigma_a^\mu).$$

We now fix the tetrad field h and let $\text{Ker } {}_h\rho$ denote the set of those elements of the group bundle Q which leave h invariant. This is a subbundle of the bundle Q . We let Σ_h denote the quotient of the bundle Q with respect to $\text{Ker } {}_h\rho$. This bundle with the coordinates $\tilde{\sigma}_a^\mu = q^\mu{}_\nu h_a^\nu$ is isomorphic to the bundle Σ equipped with the Lorentz structure of the bundle associated with $L^h X$. As a result, we can define [29], [32] the representation

$$\begin{aligned}\gamma_Q : (\Sigma_h \times T^*X) \otimes_{\Sigma_h} (\Sigma_h \times S^h) &\rightarrow (\Sigma_h \times S^h), \\ \gamma_Q : (\tilde{\sigma}, dx^\mu) &\mapsto q^\mu{}_\nu h_a^\nu \gamma^a = \tilde{\sigma}_a^\mu \gamma^a.\end{aligned}\tag{27}$$

Therefore, we can interpret the section $\tilde{h} \neq h$ of the bundle Σ_h as an effective tetrad field and $\tilde{g}^{\mu\nu} = \tilde{h}_a^\mu \tilde{h}_b^\nu \eta_{ab}$ as an effective metric. We note that \tilde{h} is not a new tetrad field and \tilde{g} is not a new metric, because the covectors $\tilde{h}^a = \tilde{h}_\mu^a dx^\mu$ are realized by γ -matrices in the same representation as the covectors $h^a = h_\mu^a dx^\mu$ and Greek indices are raised and lowered by the background metric $g^{\mu\nu} = h_a^\mu h_b^\nu \eta_{ab}$.

We thus obtain a variant of the relativistic theory of gravity (RTG) [36], [37] in the case of a background tetrad field h and a dynamic gravitational field q , which can be interpreted as a Goldstone field in the framework of gauge theory. We construct the Dirac operator and the full Lagrangian of the gauge theory with a Goldstone gravitational field.

We consider the spinor bundle S^h associated with the background tetrad field h , and spinor connection K_h given by (20) on $S^h \rightarrow X$ induced by the linear connection K given by (12) on X . Using the covariant differential D (see (21)) determined by this connection and the representation γ_Q given by (27), we can construct the Dirac operator

$$\Delta_Q = q^\lambda{}_\mu h_a^\mu \gamma^a D_\lambda.\tag{28}$$

Under restriction to the canonical section $q_0(X)$, Dirac operator (28) reduces to the Dirac operator Δ_h given by (22) on S^h for fermionic fields in the presence of a background tetrad field h and a linear connection K .

We thus obtain an extension of the RTG where the Goldstone gravitational field q , the general linear connections K , and the Dirac fermionic fields in the presence of a background field h appear as dynamic variables. The manifold of jets $J^1 Z$ of the bundle

$$Z = Q \times C \times S^h,$$

parametrized by the coordinates $(x^\lambda, q^\mu{}_\nu, k_\lambda{}^\mu{}_\nu, y^A)$ serve as the configuration space of such a model. The full Lagrangian on this configuration space can be chosen in the form of a sum

$$L = L_{AM} + L_q(q, g) + L_D$$

of the Lagrangian L_{AM} of the affine-metric theory expressed via the components of curvature tensor (26) contracted using the effective metric $\tilde{\sigma}^{\mu\nu} = \tilde{\sigma}_a^\mu \tilde{\sigma}_b^\nu \eta^{ab}$, the Lagrangian L_q of

a Goldstone gravitational field q in which contraction is performed using the background metric g , and the Dirac Lagrangian L_D given by (25) in which the tetrad gravitational field σ is replaced with the effective tetrad field $\tilde{\sigma}$. In particular, setting

$$L_{AM} = (-\lambda_1 R + \lambda_2) |\tilde{\sigma}|^{-1/2}, \quad L_q = \lambda_3 g_{\mu\nu} \tilde{\sigma}^{\mu\nu} |\sigma|^{-1/2}, \quad L_D = 0,$$

where $R = \tilde{\sigma}^{\mu\nu} R^\alpha_{\mu\alpha\nu}$, we recover the usual Lagrangian of the RTG.

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